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Null fermio-dynamics in superspace: Abelian and non-Abelian 3D massive vector fields

C Aragone†‡ and L Urrutia§

† Depto. de Física, Universidad Simón Bolívar, Apdo. 80659 Caracas, 1081A, Venezuela
§ Centro de Estudios Nucleares, UNAM, Circuito Exterior CU A Postal 70543-04510 México DF, México

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Abstract. Half-order actions, i.e. actions where each term contains, at most, one spinorial derivative, of all the different known 3D supersymmetric massive vector systems are given. Since the two real Grassmann coordinates $(\theta^1, \theta^2) \equiv \theta$ of the 3D superspace respectively constitute the two independent (fermionic) light-front projections of θ along the null bosonic directions $x^{\mp} = 2^{-1/2}(x^0 \mp x^1)$, one of them, θ^1 is regarded as the spinorial time while the other one, θ^2, is seen as a (spinorial) spacelike variable. In terms of the corresponding null spinorial timelike and spinorial spacelike derivatives the field equations contain algebraic as well as differential constraints.

We solve them for the different known 3D vector systems and obtain their corresponding unconstrained actions in superspace. In each case their evolution is shown to be controlled by a quantity which, by analogy with the standard bosonic case, is called the superenergy of the system.

The analysis of the scalar case illustrates the connection between null dynamics in superspace and the standard null dynamics in 3D bosonic spacetime. It also shows how much the superenergy contributes to the null bosonic generator of the 3D dynamics.

1. Introduction

Light-front dynamics constitute an intrinsically relativistic method of understanding the classical and quantum properties of physical systems. This approach provides a complementary picture of both the standard canonical way, where a timelike direction has to be externally picked out, and of the covariant presentation which does not show in a transparent way the dynamical structure of the system (Dirac 1949).

From the point of view of quantum field theory, quantisation along light-front coordinates is inequivalent to the canonical timelike method (Leutwyler *et al* 1970, Rohrlich and Streit 1972, Schlieder and Seiler 1972).

Light-front methods have been widely applied in the past to quantum electrodynamics (Kogut and Soper 1970), 4D Yang-Mills theory (Tomboulis 1973, Casher 1976) and classical (Aragone and Chela-Flores 1975, Scherk and Schwarz 1975, Aragone and Restuccia 1976) and quantum gravity (Kaku 1975, Kaku and Senjanovic 1977, Aragone 1978).

Recently they have successfully been used in proving the finiteness of the $N = 4$ supersymmetric Yang-Mills theories (Mandelstam 1983, Namazie *et al* 1983) in understanding their auxiliary field structure (Hassoun *et al* 1983) as well as in analysing spin-3 (and higher) fields (Bengtsson *et al* 1983).

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Here we take the point of view that the spinorial derivatives are the fundamental elements which determine the evolution of a supersymmetric system. They are the square root of the light-front bosonic derivatives. Since we regard $x^+ = 2^{-1/2}(x^0 - x^1)$ as the 'time', D_1, being the square root of $\partial/\partial x^+$, becomes the fundamental dynamical operator while D_2, the square root of $\partial/\partial x^-$, is seen as a (fermionic) spacelike operation.

In this paper we study the spinorial dynamical structure of the four presently known supersymmetric massive vector systems: the ordinary massive vector, the Abelian topological massive, the self-dual and finally the topological massive Yang–Mills system.

In each case we start from the corresponding half-order action in superspace, where each term contains, at most, one spinorial derivative. Then we single out algebraic and differential constraints and solve them. Finally we give the explicit form of the unconstrained action. As a consequence we obtain the generator of the light-front spinorial dynamics, the superenergy of the system.

The next section contains a quick review of 3D superspace and of the relevant null properties. In § 3 we consider as an example the massive scalar multiplet and the ordinary massive vector case. The scalar multiplet allows us to exhibit the connection between null fermionic superspace dynamics and the standard 3D bosonic light-front dynamics. Then, in § 4 we study two less traditional models: the Abelian topological massive vector and its close associate, the self-dual massive vector. Section 5 is dedicated to the non-Abelian topological vector system. Finally, in the last section we make some comments and discuss the results obtained.

2. Review of 3D superspace

In $D = 3$ there exist Majorana spinors $\theta = (\theta^\alpha\,;\ \alpha = 1, 2)$ and real γ matrices γ^m forming a Clifford algebra $\{\gamma^m, \gamma^n\} = 2\eta^{mn}$, $-\eta^{00} = \eta^{ii} = +1$ (i.e. $\gamma^0 \equiv \mathrm{i}\sigma_2$, $\gamma^1 \equiv \sigma_1$, $\gamma^2 \equiv \sigma_3$). They also represent the relativistic O(2, 1) Lie algebra $[2^{-1}\gamma^m\,; 2^{-1}\gamma^n] = \varepsilon^{mnp}(2^{-1}\gamma_p)$; $\varepsilon^{012} = +1$. Superfields can be expanded in its components with respect to the natural basis of the Grassmann algebra $\{1;\ \theta^\alpha;\ \theta^2 \equiv \bar\theta\theta = \bar\theta_\beta\theta^\beta = \varepsilon_{\beta\alpha}\theta^\alpha\theta^\beta;\ \varepsilon^{12} = -\varepsilon_{12} = +1\}$. Additional details can be found in Aragone (1983).

Both spinor charges and spinorial derivatives obey the anticommutation relations:

$$\{q^\alpha, \bar q_\beta\} = 2\gamma^{r\alpha}{}_\beta p_r = \{D^\alpha, \bar D_\beta\} \tag{1a}$$

where

$$\mathrm{i}q^\alpha \equiv \partial/\partial\bar\theta_\alpha - \mathrm{i}(\gamma^r\theta)^\alpha\,\partial r, \qquad D^\alpha \equiv \partial/\partial\bar\theta_\alpha + \mathrm{i}(\gamma^r\theta)^\alpha\,\partial r, \qquad p_r \equiv -\mathrm{i}\,\partial r. \tag{1b}$$

Introducing the additional momentum-like variable $p_4 \equiv 2^{-1}\bar D D$ the algebra C_3 generated by the spinorial derivatives has been shown to be equivalent to the infinite set $\{p_m{}^{\alpha_i};\ p_m{}^{\beta_i}p_4 p_m{}^{\gamma_i}(D_\alpha)^1\}$ because of the identities $D^\alpha p_4 = -p_4 D^\alpha = p^\alpha{}_\beta D^\alpha = p_r\gamma^{r\alpha}{}_\beta D^\beta$ and $p_4^2 = p_r^2$ (Aragone 1983). The product of two D can be written

$$D^\alpha D^\gamma = p^{\alpha\gamma} + p_4\varepsilon^{\alpha\gamma}. \tag{1c}$$

In 3D we choose the light-front coordinates

$$x^\pm = 2^{-1/2}(x^0 \mp x^1). \tag{2}$$

Using the representation given above for the γ matrices we have, from the right-hand side of (1):

$$D_1D_1 = -2^{-1/2}\mathrm{i}\partial_+ = 2^{-1/2}\mathbb{P}_+ \qquad D_2D_2 = -2^{-1/2}\mathrm{i}\partial_- = 2^{-1/2}\mathbb{P}_- \tag{3a}$$

$$D_1D_2 + D_2D_1 = -2\mathrm{i}\partial_2 = 2\mathbb{P}_2. \tag{3b}$$

They make evident the square root connection between the fermionic operators $(D_1; D_2)$ and the light-front evolution operators (∂_+, ∂_-). θ^1 is the fermionic time associated with x^+ while θ^2 is the fermionic spacelike coordinate due to x^-.

Let us consider as an example a (real) massive scalar superfield $\phi(x, \theta) \equiv a(x) + \mathrm{i}\bar{\theta}_\alpha \psi^\alpha(x) + \mathrm{i}\theta^2 f(x)$.

Its components can be obtained by successive spinorial derivatives taken at vanishing θ (Gates *et al* 1983):

$$\phi|_{\theta=0} = a(x) \qquad D^\alpha \phi| = \mathrm{i}\psi^\alpha(x) \qquad p_4\phi| = -2\mathrm{i}f(x). \tag{4a}$$

The 3D component action can be obtained from its superspace form by reducing the integration in the θ form to spinorial derivatives at vanishing θ:

$$\langle HK \rangle \equiv \int \mathrm{d}^2\theta\, \mathrm{d}^3x\, H(x, \theta)K(x, \theta) = -\int \{2p_4\{H(x, \theta)K(x, \theta)\}\}\, \mathrm{d}^3x. \tag{4b}$$

For instance, the action for the scalar massive multiplet (in superspace (x, θ)) is

$$I = 2^{-1/2}\langle \phi p_4 \phi + \mathrm{i}m\phi^2 \rangle. \tag{5a}$$

Its component form will be

$$\begin{aligned} I &= -\langle p_4\phi| \cdot p_4\phi| + \phi| \cdot (p_r^2\phi)| + D_\alpha\phi| \cdot D^\alpha p_4\phi| \\ &\qquad + 2\mathrm{i}m(p_4\phi)| \cdot \phi| + \mathrm{i}mD_\alpha\phi| \cdot D^\alpha\phi|\rangle \\ &= \langle a \cdot \Box a - 4maf + 4f^2 - \mathrm{i}\bar{\psi}\partial\!\!\!/ \psi + \mathrm{i}m\bar{\psi}\psi \rangle. \end{aligned} \tag{5b}$$

The vector supermultiplet is carried by a Majorana superspinor $\Psi^\alpha(x, \theta)$

$$\Psi^\alpha(x, \theta) \equiv \psi^\alpha(x) + \bar{\theta}_\beta\{\tfrac{1}{2}v\varepsilon^{\beta\alpha} + v^r\gamma_r{}^{\beta\alpha}\} + \mathrm{i}\theta^2\phi^\alpha(x) \tag{6a}$$

whose components can be obtained in a way similar to (4*a*):

$$\Psi^\alpha| = \psi^\alpha(x) \qquad D^\beta\Psi_\alpha| = -\tfrac{1}{2}v\delta^\beta_\alpha + v^r\gamma_r{}^\beta{}_\alpha \qquad p_4\Psi^\alpha| = -2\mathrm{i}\phi^\alpha(x). \tag{6b}$$

These components fields, regarded as spinorial derivatives at vanishing θ, will be very useful when evaluating the respective 3D action using (4*b*) and the properties mentioned earlier (1*c*).

We can now understand why each half-order action gives rise, when θ-integrated, to the corresponding first-order one in the 3D effective world. By definition

$$I^{1/2} = \sum \int \mathrm{d}^2\theta\, \mathrm{d}^3x\, H^{\alpha A} D_\alpha K_A. \tag{7a}$$

Application of (4*b*) tells us that

$$\begin{aligned} \int \mathrm{d}^2\vartheta\, \mathrm{d}^3x\, H^{\alpha A} D_\alpha K_A &= -\int 2p_4(H^{\alpha A} D_\alpha K_A)|(\mathrm{d}^3x) \\ &= -\int \mathrm{d}^3x\{2p_4 H^{\alpha A}| \cdot D_\alpha K_A| + H^{\alpha A}| \cdot 2p_4 D_\alpha K_A \\ &\qquad + (-1)^{a(A)+1} \cdot 2D_\beta H^{\alpha A}| \cdot D^\beta D_\alpha K_A|\}. \end{aligned} \tag{7b}$$

$2p_4H^{\alpha A}|$, $D_\alpha K_A|$, $H^{\alpha A}|$ and $D_\beta H^{\alpha A}|$ are component fields of $H^{\alpha A}$ or K_A, according to definitions similar to (6*b*). Moreover $p_4 D_\alpha K| = p_\alpha{}^\beta D_\beta K| = -\mathrm{i}\partial_\alpha{}^\beta(D_\beta K_A)|$ brings in one derivative and $D^\beta D_\alpha K_A| = (p^\beta{}_\alpha K_A)| - (p_4\delta^\beta_\alpha K_A)| = -\mathrm{i}\partial_\alpha{}^\beta(K_A)| - \delta^\beta_\alpha(p_4 K_A)|$ introduces another derivative in the last term of (7*b*). Therefore the component action (7*b*) is just of the first-order type, the terms with one derivative being $H^{\alpha A}| \cdot (-\mathrm{i}\partial^\beta_\alpha(D_\beta K_A)|$ and $(-\mathrm{i}D_\beta H^{\alpha A})| \cdot \partial_\alpha{}^\beta(K_a)|$.

3. The standard scalar and vector massive supermultiplets

The half-order action of the massive scalar supermultiplet is given by

$$I = \tfrac{1}{2}\langle p_\alpha D^\alpha \phi - 2^{-1} \mathrm{i} p_\alpha p^\alpha + m\phi^2 \rangle \tag{8a}$$

where $\langle \ldots \rangle$ stands for integration with respect to $\mathrm{d}^5 z \equiv (\mathrm{d}^3 x)(\mathrm{d}^2\theta)$.

Independent variations with respect to p^α and ϕ yield

$$\mathrm{i}p^\alpha = D^\alpha \phi \qquad -\tfrac{1}{2} D_\alpha p^\alpha + m\phi = 0. \tag{9a, b}$$

After insertion of p^α into $(5b)$ one obtains the right first-order supersymmetric equation stating the massive propagation of ϕ: $(\mathrm{i}p_4 + m)\phi = 0$.

Expanding this action in terms of (p^α, ϕ) we have

$$I = \tfrac{1}{2}\langle p^2(-D_2\phi + \mathrm{i}p^1) - p^1(D_1\phi) + m\phi^2 \rangle \tag{8b}$$

where we see that p^2 is the multiplier associated with the algebraic constraint $C_2 \equiv \mathrm{i}p^1 - D_2\phi = 0$ (no D_1 or ∂_+ derivatives in it). After solving it we get the unconstrained null spinorial formulation of this action:

$$I = 2^{-1}\langle D_2\phi \cdot \mathrm{i}D_1\phi + m\phi^2 \rangle \cong 2^{-1}\langle \phi \cdot \mathrm{i}p_4\phi + \phi m\phi \rangle. \tag{8c}$$

We observe the similar structure of this action compared with the standard form of bosonic actions in the light-front coordinates which has the form $\langle \dot{q}q' - J(q, q')\rangle$. Here $D_2 \sim (\partial_-)^{1/2}$ and $D_1 \sim (\partial_+)^{1/2}$ and the role of the null energy, which is a non-conserved non-negative quantity, has been taken by $\langle m\phi^2 \rangle$.

On the field equations the superenergy $\langle m\phi^2\rangle \sim \langle 4f^2 - 2^{-1} m \mathrm{i}\psi^2 \rangle$ which is a real quantity of the same dimensions as the energy, having no definite sign.

Integration of $(8a)$ with respect to the θ variables yields the first-order form of the 3D component action, using the expansion $(7b)$.

The real Majorana superspinor p^α has the form:

$$p^\alpha(x, \theta) = p^\alpha(x) + \bar{\theta}_\beta\{\tfrac{1}{2}\varepsilon^{\beta\alpha} p(x) + q^r \gamma_r{}^{\beta\alpha}\} + \mathrm{i}\theta^2 \pi^\alpha. \tag{10}$$

Its components can be recovered through spinorial derivatives at vanishing θ, as shown in $(6b)$. After insertion of the different component fields into the right-hand side of $(7b)$ one finds (up to an overall i)

$$I = \langle q_r^2 + 2q_r \partial_r a - \tfrac{1}{4}p^2 + 4maf - 2pf - \mathrm{i}\bar{p}\partial\!\!\!/ \psi - 2\mathrm{i}\bar{\pi}p + 2\bar{\pi}\psi - \mathrm{i}m\bar{\psi}\psi \rangle. \tag{11}$$

The θ integration can also be done after one has reached the unconstrained null spinorial formulation $(8c)$, using $(1c)$, $(3a)$, and taking into account that the null γ matrices $\gamma^\pm \equiv 2^{-1/2}(\gamma^0 \mp \gamma^1)$ have the respective values:

$$\gamma^+ = -2^{1/2}\begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \end{pmatrix} \qquad \gamma^- = 2^{1/2}\begin{pmatrix} \cdot & 1 \\ \cdot & \cdot \end{pmatrix}. \tag{12a}$$

We get (up to an overall $\tfrac{1}{2}\mathrm{i}$, $\partial f/\partial x^+ \equiv \dot{f}$, $\partial f/\partial x^- \equiv f'$)

$$\begin{aligned} I &= \langle \tfrac{1}{2}a'\dot{a} + 4mfa + 4f^2 - (\partial_z a)^2 - \mathrm{i}\bar{\psi}\partial\!\!\!/\psi - \mathrm{i}m\bar{\psi}\psi \rangle \\ &= \langle \tfrac{1}{2}a'\dot{a} + 4mfa - 4f^2 - (\partial_2 a)^2 + \sqrt{2}\mathrm{i}\psi^1\dot{\psi}^1 + \sqrt{2}\mathrm{i}\psi^2\psi^{2'} \\ &\quad + 2\mathrm{i}\psi^2\partial_2\psi^1 - 2\mathrm{i}m\psi^2\psi^1 \rangle. \end{aligned} \tag{12b}$$

The generator of the null dynamics contains two types of contributions: the first is the θ integration of the superenergy and the second is of terms $\sim D_1 D_2\phi| \cdot D_2 D_1\phi|$ stemming from the $(7b)$ expansion of the null spinorial dynamical term $\int \mathrm{i}D_2\phi \cdot D_1\phi(\mathrm{d}^3x)\,\mathrm{d}^2\theta$ of $(8c)$.

The basic superfield needed to represent a vector system is a Majorana superspinor $\Psi^\alpha(x,\theta)=\Psi^\alpha(x,\theta)^+$ as defined in (6*a*).

The half-order action of the supersymmetric ordinary 3D massive vector system is

$$I_{st}=\tfrac{1}{2}\langle \mathrm{i}p_{\alpha\beta}D^\beta p^\alpha-\mathrm{i}q^{\alpha\beta}D_\beta\Psi_\alpha-p_{\alpha\beta}q^{\beta\alpha}+p_\alpha p^\alpha-m^2\Psi_\alpha\Psi^\alpha\rangle \tag{13a}$$

where the independent variables are $\{p^{\alpha\beta};q^{\alpha\beta};p_\alpha;\Psi_\beta\}$. In terms of the superstrength $W^\alpha=\mathrm{i}D_\beta D^\alpha\Psi^\beta$ (13*a*) becomes

$$I_{st}=2^{-1/2}\langle\tfrac{1}{4}W_\alpha W^\alpha-m^2\Psi_\alpha\Psi^\alpha\rangle \tag{13b}$$

which can be integrated out in its θ variables in order to obtain the familiar 3D component form. It turns out to be

$$I_{st}=\langle 4f_r^2(v_s)-m^2v_r^2+\tfrac{1}{4}m^2v^2-\mathrm{i}(\bar\phi+2^{-1}\overline{\partial\!\!\!/\psi})\partial\!\!\!/(\phi+2^{-1}\partial\!\!\!/\psi)+2\mathrm{i}m^2\bar\psi\phi\rangle \tag{13c}$$

where we are using the dual field strength $f_r(v_s)\equiv\varepsilon_r^{st}\partial_s v_t$.

After expanding the action (13*a*) three algebraic constraints can be solved, allowing us to eliminate their corresponding multipliers (p^{12}, p^{22}, q^{22}). Its reduced form is

$$\begin{aligned}I_{st,red}=2^{-1}\langle &D_2\Psi_2\cdot D_1p_1+\mathrm{i}p^{21}D_1p_2-\mathrm{i}q^{21}D_1\Psi_2+D_2p_2\cdot D_1\Psi_1\\&-D_2p_1\cdot D_2\Psi_2+q^{21}\cdot p^{21}+2p_1p_2-2m^2\psi_1\Psi_2\rangle.\end{aligned} \tag{14a}$$

In this action it is convenient to introduce two new variables (p,q) to replace p^{21} and q^{21}:

$$p^{21}\equiv \mathrm{i}p-\mathrm{i}D_2\Psi_1\qquad q^{21}\equiv q+\mathrm{i}D_2p_1. \tag{15}$$

Two differential constraints appear, associated with Ψ_1 and p_1 as multipliers. The new form of the action reads

$$\begin{aligned}I_{st,red}=\tfrac{1}{2}\langle &p_1(2\mathbb{P}_2\Psi_2+2p_2-D_2p)+\Psi_1(2\mathbb{P}_2p_2-2m^2\Psi_2-\mathrm{i}D_2q)\\&-p\cdot D_1p_2-\mathrm{i}qD_1\Psi_2+\mathrm{i}p\cdot q\rangle.\end{aligned} \tag{14b}$$

It is possible to solve the two differential constraints:

$$\begin{aligned}&2\mathbb{P}_2\Psi_2+2p_2=D_2p\\&-2m^2\Psi_2+2\mathbb{P}_2p_2=\mathrm{i}D_2q\end{aligned} \tag{16a, b}$$

in terms of the new variables p, q. Calling $\Delta_m\equiv\mathbb{P}_2^2+m^2=-\partial_{22}+m^2$ the massive non-negative Laplace operator one gets for (Ψ_2,p_2)

$$\begin{aligned}&\Psi_2=-\tfrac{1}{2}\mathrm{i}\Delta_m^{-1}D_2(q+\mathrm{i}\mathbb{P}_2p)\\&p_2=\tfrac{1}{2}\mathrm{i}\Delta_m^{-1}D_2(\mathbb{P}_2q-\mathrm{i}m^2p).\end{aligned} \tag{17a, b}$$

After their insertion in (8*b*) the unconstrained form of the action is found:

$$I_{st,unc}=\tfrac{1}{4}\langle -D_2p\cdot(m^2/\Delta_m)D_1p-D_2q\cdot(1/\Delta_m)D_1q+\mathrm{i}p\cdot(2m^2/\Delta m)q\rangle. \tag{18a}$$

Observe that it essentially has the typical null canonical structure. Redefining (p,q)

$$p\Rightarrow m\Delta_m^{-1/2}p\qquad q\Rightarrow\Delta_m^{-1/2}q$$

the unconstrained action becomes

$$I_{alg,unc}=4^{-1}\mathrm{i}\langle p\cdot \mathrm{i}p_4p+q\mathrm{i}p_4q+2pmq\rangle. \tag{18b}$$

This action depends on two independent supersymmetric pseudo-scalar real variables, as it must, the reason being that this system is a massive parity-invariant vector.

In $D=3$ it has to contain the two $s=+1$ and $s=-1$ pseudo-scalars plus their respective supersymmetric partners (Deser *et al* 1982). The dynamics takes a very simple form:

$$\mathrm{i}p_4 p + mq = 0 \qquad \mathrm{i}p_4 q + mp = 0. \tag{19a}$$

In terms of (w_1, w_2) given by $p \pm q = 2^{-1} w_{1,2}$ this action takes the final uncoupled form:

$$I_{\mathrm{st,unc}} = \tfrac{1}{8}\mathrm{i}\langle w_1 \cdot \mathrm{i}p_4 w_1 + w_2 \cdot \mathrm{i}p_4 w_2 + m w_1^2 - m w_2^2\rangle. \tag{18c}$$

Both w_1 and w_2 obey the scalar massive propagation equation:

$$(\mathrm{i}p_4 \pm m) w_{1,2} = 0. \tag{19b}$$

In terms of the component fields of w_1 and w_2, $w_i \equiv a_i + \mathrm{i}\bar{\theta}\psi_i + \mathrm{i}\theta^2 f_1$, $i=(1,2)$, the superenergy takes the real value

$$\langle m w_1^2 - m w_2^2\rangle \sim \langle 4 f_1^2 - 2^{-1} m \mathrm{i}\psi_1^2 - 4 f_2^2 + 2^{-1} m \mathrm{i}\psi_2^2\rangle.$$

It is also worth noticing that, when the mass goes to zero, the unconstrained action ($12c$) goes to the addition of two decoupled massless scalars, since massless 3D excitations do not have helicity (Binegar 1982).

4. The Abelian topological massive vector system and the self-dual model

A convenient form of the gauge invariant (under the Abelian gauge transformation $\delta\Psi_\alpha = \mathrm{i}^{-1} D_\alpha \zeta$) half-order action for the Abelian topological massive supersymmetric vector system is

$$I_m^{\mathrm{top}} = \tfrac{1}{2}\langle \mathrm{i}^{-1} q^{\alpha\beta} D_\beta \Psi_\alpha - \tfrac{1}{2} p_{\alpha\beta} D^\beta W^\alpha - \tfrac{1}{4} W_\alpha W^\alpha - p_{\alpha\beta} q^{\beta\alpha} + \tfrac{1}{2}\mathrm{i} m p_{\alpha\beta} p^{\beta\alpha}\rangle \tag{20}$$

which, when $m \to 0$, goes over the Abelian massless case (($13a$) for $m=0$) with $p^\alpha = 2^{-1}\mathrm{i} W^\alpha$.

Independent variations of the four layers of independent covariant superfields yield

$$p_{\alpha\beta} = \mathrm{i}^{-1} D_\alpha \Psi_\beta \qquad W^\alpha = -D_\beta p^{\alpha\beta} \tag{21a}$$

$$q^{\alpha\beta} = -\tfrac{1}{2} D^\alpha W^\beta + \mathrm{i} m p^{\alpha\beta} \qquad D_\beta q^{\alpha\beta} = 0. \tag{21b}$$

Expanding the action (14) in terms of the different spinorial components $(p^{11}, p^{12}, p^{21}, p^{22}, W^1, W^2, \ldots,)$ three algebraic constraints can be recognised:

$$p^{11} = -\mathrm{i} D_2 \Psi_2 \qquad p^{12} = \mathrm{i} D_2 \Psi_1 \qquad q^{11} = \mathrm{i} m p^{11} - \tfrac{1}{2} D_2 W_2. \tag{22}$$

Their respective multipliers are p^{22}, q^{22} and q^{12}.

After introducing these values of p^{11}, p^{12} and q^{11} into the initial action, it can be seen that, in the reduced action, (Ψ_1, W_1) constitute an additional set of Lagrange multipliers associated with two differential constraints:

$$\mathrm{i}\mathbb{P}_2 \Psi_2 - \tfrac{1}{2} W_2 = \tfrac{1}{2}\mathrm{i} D_2 p \tag{23a}$$

$$-2m\mathbb{P}_2 \Psi_2 + \mathbb{P}_2 W_2 = D_2 q. \tag{23b}$$

(Two new variables $\mathrm{i}p \equiv p^{21} + \mathrm{i} D_2 \Psi_1$; $q \equiv q^{21} + 2^{-1} D_2 W_1 - m D_2 \Psi_1$ have been defined.)

Their solution can be expressed in the form $(\Delta_0 \equiv \Delta_m|_{m=0} = \mathbb{P}_2^2)$

$$\Psi_2 = -\tfrac{1}{2}\mathrm{i} D_2 \Delta_0^{-1} \Delta_m^{-1} \mathbb{P}_2 (\mathbb{P}_2 - \mathrm{i}m)(q + \mathrm{i}\mathbb{P}_2 p) \tag{24a}$$

$$W_2 = D_2 \Delta_m^{-1} (\mathbb{P}_2 - \mathrm{i}m)(q + mp). \tag{24b}$$

Insertion of these values of Ψ_2, W_2 and use of the identity $D_1D_2 \equiv \mathbb{P}_2 + p_4$ into the reduced form of the action (14) leads to the unconstrained action:

$$I^{\mathrm{top}}_{m,\mathrm{unc}} = \tfrac{1}{4}\mathrm{i}\langle pm^2 \cdot (1/\Delta_m)(\mathrm{i}p_4 + m)p + q \cdot (1/\Delta_m)(\mathrm{i}p_4 + m)q + 2p \cdot m(1/\Delta_m)(\mathrm{i}p_4 + m)q\rangle$$
$$\equiv \tfrac{1}{4}\mathrm{i}\langle w \cdot (\mathrm{i}p_4 + m)w\rangle \qquad (25)$$

where we have been able to cast $I^{\mathrm{top}}_{m,\mathrm{unc}}$ as the functional of a unique variable $w = \Delta_m^{-1/2}(q + mp)$. This is what one must expect on physical grounds, since the topological model is a parity-sensitive theory carrying either spin $+1$ (or -1) plus the necessary supersymmetric fermionic partner, according to the sign behind the mass in the last term of the action (14). The final expression of (25) shows that the Abelian topological model is just the w_1 part of the ordinary massive theory, as explicitly exhibited in (18c). It is also worth pointing out that the final physical variable $w = 2^{-1}D_1\Delta_m^{-1/2}W_2 + 2^{-1}D_2\Delta_m^{-1/2}W_1$ is a gauge independent quantity.

The self-dual bosonic model (Townsend *et al* 1984, Deser and Jackiw 1984) can be supersymmetrised and formulated in superspace. It is sufficient to notice that the propagation equation of the set (21) can be written in the form:

$$D_\beta D^\alpha\{W^\beta - 2m\Psi^\beta\} = 0. \qquad (26)$$

Locally this is equivalent to asserting that $W^\beta - 2m\Psi^\beta$ is a pure gauge, i.e.

$$W^\beta - 2m\Psi^\beta = \mathrm{i}^{-1}D^\beta\xi \qquad \xi = \xi^+. \qquad (27)$$

Therefore, if one chooses a Lorentz supergauge, this equation can be reduced to $W^\beta - 2m\Psi^\beta = 0$ where, of course, Ψ^β must now be transverse $D_\beta\Psi^\beta = 0$ because of the Bianchi identity on W^β. The half-order self-dual action generating these field equations has the superspace form:

$$I^{\text{self-dual}}_m = \mathrm{i}m\langle p^{\alpha\beta}D_\beta\Psi_\alpha - p^{\alpha\beta}p_{\beta\alpha} + \tfrac{1}{2}\mathrm{i}\Psi_\alpha m\Psi^\alpha\rangle. \qquad (28a)$$

In terms of the superstrength W^α introduced before (13b) it can be written as

$$I^{\text{self-dual}} = m\langle\tfrac{1}{4}\Psi_\beta W^\beta - \tfrac{1}{2}m\Psi_\alpha\Psi^\alpha\rangle \qquad (28b)$$

which, after θ integration, becomes the supersymmetric version of the self-dual model:

$$I^{\text{self-dual}} = 2m\langle -v_r f_r(v_s) - \tfrac{1}{2}mv_r^2 + \tfrac{1}{8}mv^2 + \mathrm{i}(\bar{\phi} + \tfrac{1}{2}\overline{\not{\partial}\psi})(\phi + \tfrac{1}{2}\not{\partial}\psi) - 2\mathrm{i}m\bar{\phi}\psi\rangle. \qquad (28c)$$

Independent variations of Ψ_α, $p^{\alpha\beta}$ in (28a) lead to

$$p_{\alpha\beta} = \tfrac{1}{2}D_\alpha\Psi_\beta \qquad D_\beta p^{\alpha\beta} + \mathrm{i}m\Psi^\alpha = 0. \qquad (29)$$

The $1+1$ null expansion of (22) according to the independent spinorial components (p^{11}, p^{12}, p^{21}, p^{22}, Ψ_1, Ψ_2) explicitly shows the presence of two algebraic constraints:

$$p^{11} = \tfrac{1}{2}D_2\Psi_2 \qquad p^{12} = -\tfrac{1}{2}D_2\Psi_1 \qquad (30)$$

the former associated to p^{22} at its multiplier, the latter (quadratic) arising from the terms containing p^{12}. After we substitute these values of p^{11} and p^{12} into the expanded action we get a reduced form of (22), depending upon the three (super)variables (p^{21}, Ψ_α):

$$I^{\text{self-dual}}_{m,\mathrm{red}} = \mathrm{i}m\langle\tfrac{1}{2}(D_2\Psi_2)\cdot(D_1\Psi_1) - \tfrac{1}{4}(D_2\Psi_1)(D_2\Psi_1) + (p^{21})^2$$
$$+ p^{21}\cdot D_1\Psi_2 + \mathrm{i}m\Psi_1\Psi_2\rangle. \qquad (31)$$

The first term can be transformed (taking into account (3b)) into

$$\tfrac{1}{2}\langle D_2\Psi_2 \cdot D_1\Psi_1\rangle \cong \langle\tfrac{1}{2}\Psi_2 \cdot D_2D_1\Psi_1\rangle = -\langle\tfrac{1}{2}\Psi_2 D_1D_2\Psi_1\rangle$$
$$+ \langle\Psi_1 \cdot \mathbb{P}_2\Psi_2\rangle \cong -\tfrac{1}{2}\langle D_1\Psi_2 \cdot D_2\Psi_1\rangle + \langle\Psi_1 \cdot \mathbb{P}_2\Psi_2\rangle.$$

The new form of the action (31) suggests shifting from p^{21} to the new variable $p \equiv p^{21} - 2^{-1/2} D_2 \Psi_1$.

In this way an additional differential constraint appears having Ψ_1 as its associated multiplier:

$$I^{\text{self-dual}}_{m,\text{red}} = \mathrm{i}m\langle \Psi_1 \cdot \{(\mathbb{P}_2 + \mathrm{i}m)\Psi_2 + D_2 p\} + p \cdot D_1 \Psi_2 + p^2 \rangle. \tag{32a}$$

The solution of this last constraint is

$$\Psi^2 = -\Delta_m^{-1}(\mathbb{P}_2 - \mathrm{i}m) D_2 p. \tag{32b}$$

Once (32*b*) is introduced into $I^{\text{self-dual}}_{m,\text{red}}$, we have the unconstrained action of this supersymmetric self-dual model in terms of the unique dynamical variable p one is left with (corresponding to the fact that this parity-sensitive model must only contain the supersymmetric extension of the unique physical component of the bosonic self-dual model carrying either spin $+1$ or -1). The unconstrained self-dual action is

$$I^{\text{self-dual}}_{m,\text{unc}} = \langle p \cdot (m^2/\Delta_m)(\mathrm{i}p_4 + m)p \rangle = \langle w \cdot (\mathrm{i}p_4 + m) w \rangle \tag{33}$$

once the redefinition $w \equiv m\Delta_m^{-1/2} \cdot p$ has been introduced. The self-dual and topological models are equivalent, as is shown by the respective last terms of (25) and (33).

5. The super-non-Abelian topological massive vector system

The basic superpotential is given by $\Psi^\alpha \equiv g\Psi^{\alpha a} T_a$ where $T_a^+ = T_a$ are the Hermitian generators of the internal semisimple group, $[T_a, T_b] = \mathrm{i}c_{abc}T_c$, $\Psi^{\alpha a}$ are Majorana superspinors and the 3D coupling constant $g \sim M^{1/2}$.

The initial formulation of this system was given by Schonfeld (1981). Thereafter, alternative improved superspace derivations were independently given by Aragone (1983) and Gates *et al* (1983). The infinitesimal inhomogeneous transformation law of Ψ^α is

$$\delta_\omega \Psi^\alpha = -\mathrm{i}\mathscr{D}^\alpha \omega \qquad \mathscr{D}^\alpha \omega \equiv D^\alpha \omega - (\Psi^\alpha, \omega). \tag{34}$$

(Round brackets are assumed to be either commutators or anticommutators, according to whether one of the quantities involved is a boson or whether both are fermions.)

It is also useful to introduce the operator $\mathscr{D}'^\alpha(\cdot) \equiv D^\alpha(\cdot) - 2^{-1}(\Psi^\alpha, \cdot)$. It allows a compact expression both of the gauge covariant superfield strength

$$W^\alpha \equiv \mathrm{i}\mathscr{D}'_\beta \mathscr{D}'^\alpha \Psi^\beta + \tfrac{1}{12}\mathrm{i}(\Psi_\beta(\Psi^\alpha, \Psi^\beta)) \tag{35a}$$

and of the distorted superfield strength

$$\boldsymbol{W}^\alpha \equiv W^\alpha + \tfrac{1}{3}\mathrm{i}(\Psi_\beta, \mathscr{D}'^\beta \Psi^\alpha). \tag{35b}$$

The half-order action of this non-Abelian system is given by

$$\begin{aligned} I^{\text{top}} = \frac{1}{2g^2}\langle & \mathrm{i}^{-1} q^{\beta\alpha} \mathscr{D}'_\alpha \Psi_\beta - \tfrac{1}{2} p^{\alpha\beta} \mathscr{D}'_\beta \boldsymbol{W}_\alpha + \tfrac{1}{24}\mathrm{i}\, \boldsymbol{W}_\alpha \cdot (\Psi_\beta(\Psi^\alpha, \Psi^\beta)) \\ & - \tfrac{1}{4} \boldsymbol{W}_\alpha \boldsymbol{W}^\alpha - p^{\alpha\beta} \cdot q_{\beta\alpha} + \tfrac{1}{2}\mathrm{i}m p_{\alpha\beta} p^{\beta\alpha} + \tfrac{1}{6} m p^{\alpha\beta} \cdot (\Psi_\alpha, \Psi_\beta) \\ & - \tfrac{1}{24}\mathrm{i}m \Psi_\alpha \cdot (\Psi_\beta(\Psi^\alpha, \Psi^\beta))\rangle \end{aligned} \tag{36}$$

which is a generalisation, for the non-Abelian case, of (20). (Independent variations of $q^{\beta\alpha}$, $p^{\beta\alpha}$, W^α yield

$$p_{\alpha\beta}=-\mathrm{i}\mathscr{D}'_\alpha\Psi_\beta \qquad W^\alpha=-\mathscr{D}'_\beta p^{\alpha\beta}+\tfrac{1}{12}\mathrm{i}(\Psi_\beta(\Psi^\alpha,\Psi^\beta)) \tag{37a, b}$$

$$q^{\alpha\beta}=-\tfrac{1}{2}\mathscr{D}'^\alpha W^\beta+\mathrm{i}mp^{\alpha\beta}+\tfrac{1}{6}m(\Psi^\alpha,\Psi^\beta) \tag{37c}$$

which inserted into (32) lead to the second-order action $4^{-1}g^{-2}\langle 2^{-1}W^2-m\Psi\cdot W\rangle$.)

We first look for the algebraic constraints in the $1+1$ expansion of the half-order action (36). Calculations become simpler in the supersymmetric algebraic gauge $\Psi_2=0$ that we choose from now on.

In the first step three linear constraints are singled out, their solutions being

$$q^{11}=-\tfrac{1}{2}\mathscr{D}'_2W_2 \qquad p^{11}=0 \qquad p^{12}=\mathrm{i}\mathscr{D}'_2\Psi_1. \tag{38}$$

They appear associated respectively with $(p^{22};q^{22};q^{12})$ as Lagrange multipliers. After we introduce these solutions (38) in the initial action (36) we obtain the reduced expression:

$$I^{\text{top}}_{\text{red}}=\frac{1}{2g^2}\langle -\tfrac{1}{2}p^{21}\cdot\mathscr{D}'_1W_2+\tfrac{1}{2}\mathrm{i}D_2W_2\cdot\mathscr{D}'_1\Psi_1+\tfrac{1}{2}\mathrm{i}m(D_2\Psi_1)(D_2\Psi_1)$$

$$+p^{21}\cdot q^{21}-\tfrac{1}{2}W_1\cdot(W_2+\mathrm{i}D_2D_2\Psi_1)-\tfrac{1}{2}\mathrm{i}mp^{21}\cdot p^{21}\rangle \tag{39}$$

and two new algebraic constraints emerge:

$$p^{21}=0 \qquad W_2+\mathrm{i}D_2D_2\Psi_1\equiv W_2+2^{-1/2}\Psi'_1=0. \tag{40a, b}$$

Substitution of these values of (p^{21}, W_2) into the previously given reduced form provides the unconstrained action as a functional of the unique fermionic supercomponent Ψ_1:

$$I^{\text{top}}_{\text{red,unc}}=\frac{1}{2^{3/2}g^2}\langle\tfrac{1}{2}\Psi'_1\cdot(\mathrm{i}p_4+m)\Psi_1+\tfrac{1}{2}\mathrm{i}(\Psi'_1,D_2\Psi_1,\Psi_1)\rangle. \tag{41a}$$

The corresponding field equation has the form:

$$(\mathrm{i}p_4+m)\Psi'_1+\mathrm{i}(D_2\Psi'_1,\Psi_1)+\tfrac{1}{2}\mathrm{i}(D_2\Psi_1,\Psi'_1)=0 \tag{41b}$$

which is also meaningful when the mass goes to zero. Even in the massless case, the 3D super Yang–Mills determines a non-trivially self-coupled theory, as the unconstrained action demonstrates.

Observe that the reduced action owes its dynamical non-triviality to the presence of a cubic self-interacting term.

This cubic term is the one which prevents writing the unconstrained action in terms of a supervariable like $D_2\Psi_1\sim w$. The superenergy is given in this case by

$$\langle\tfrac{1}{2}\Psi'_1\cdot m\Psi_1+\tfrac{1}{2}\mathrm{i}(\Psi'_1,D_2\Psi_1,\Psi_1)\rangle. \tag{42}$$

It is also worth pointing out that, since the gauge choice $\Psi_2=0$ is an algebraic one, and since we have reached the unconstrained action just by solving algebraic constraints, then when computing the effective action in this sypersymmetric null gauge there will not be ghosts in it, not to say that the vertex structure stemming from the unique cubic term $\sim(\Psi'_1,D_2\Psi_1,\Psi_1)$ is very simple.

6. Discussion

We have shown that if one takes the evolution along the Grassmann coordinates as the basic evolution operators generating the whole dynamics of a (vector) supersymmetric system they naturally induce a supersymmetric bosonic time $(-i\partial_4 \sim p_4)$ along which the system evolves in a first-order way. The unconstrained actions for the free massive vector system, as well as for the Abelian topological massive and self-dual model (which are equivalent for non-vanishing mass), have been found in terms of the expected number of supercomponents: 2 in the first case due to its parity invariant structure, and 1 for the latter ones which, being parity sensitive, contains one definite value of the spin of the propagating vector (either +1 or −1).

In addition to these free systems we have also analysed both the massless and the topological massive vectorial non-Abelian supersymmetric systems. A convenient algebraic gauge $\Psi_2 = 0$ was found which permits the rapid identification of all the constraints these systems have. Since they are all algebraic, one may solve them and obtain the very simple form given in (37a) where it can be seen that interactions are due to the existence of a cubic contribution to the superspace action.

In this intrinsically supersymmetric picture there is a quantity which generates the dynamics. This quantity is what we have defined as the superenergy of the system. For all the vector systems we have treated their corresponding supersymmetric covariant half-order action was given. These results demonstrate that the basic propagator of the unique fermionic supervariable Ψ_1 is essentially $(ip_4 + m)^{-1} = (-p^2 + m^2)^{-1} \cdot (ip_4 - m)$ and that the effective action is ghost free.

In four dimensions the Grassmann algebra becomes complex (Aragone 1985) and consequently there are two spinorial time derivatives $(D_1, \bar{D}_1)$ one has to consider as the fundamental evolution operators. This is a technical problem that makes calculations lengthier but does not prevent us from performing a similar analysis leading to the unconstrained formulation of supersymmetric systems.

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